Dealing with some interesting formulations of Maxwell's Equations

Rohan Biswas
Dept. of Computer Science and Engineering ,
Tezpur University,India

January 11, 2024

Abstract

Maxwell's equations, fundamental in the field of electromagnetism, are often expressed in various formulations to provide different insights into the underlying physical principles. This paper explores and compares different formulations of Maxwell's equations, including the vector algebraic differential form, potential form, 4-vector form, and differential geometric form. By delving into these different representations, we aim to deepen our understanding of the profound nature of Maxwell's equations and their significance in describing the fundamental principles governing electromagnetic phenomena. Only basic knowledge of linear algebra is assumed.

1 Introduction

Let us start with Maxwell's equations in a charge-free vacuum. Now, let the electric field be $\mathbf{E}$=(t, E_x, E_y, E_z) and the magnetic field be $\mathbf{B}$=(t, B_x, B_y, B_z) . These are the two vectors that we are gonna be dealing with throughout this paper.

The first equation, also known as Gauss law of Electric Field in charge-free vacuum is:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{1}$$

The second equation, also known as Gauss law of Magnetic field is:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{2}$$

The third equation, or Faraday's law, can be shown in three spatial coordinates (x, y, z) as:

$$\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = \frac{\partial B_x}{\partial t}, \tag{3a}$$

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = \frac{\partial B_y}{\partial t}, \tag{3b}$$

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = \frac{\partial B_z}{\partial t}. \tag{3c}$$

The fourth equation , also known as Ampere's law , for charge -free vacuum can be shown in three spatial coordinates (x, y, z) as :

$$\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} = -\frac{\partial E_x}{\partial t}, \tag{4a}$$

$$\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} = -\frac{\partial E_y}{\partial t}, \tag{4b}$$

$$\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = -\frac{\partial E_z}{\partial t}. \tag{4c}$$

Now that we have seen how these equations look in charge-free equations , it's time to introduce to some weird yet fascinating ways of rewriting these equations.Like the potential form :

$$\nabla^2 \mathrm{V} + \frac{\partial}{\partial t}(\nabla \cdot \mathrm{A}) = -\frac{\rho}{\epsilon_0}, \tag{5a}$$

$$\nabla \cdot (\nabla \times \mathrm{A}) = 0, \tag{5b}$$

$$-\nabla^2 \mathrm{A} + \epsilon_0 \mu_0 \frac{\partial^2 \mathrm{A}}{\partial t^2} = -\nabla\Big(\nabla \cdot \mathrm{A} + \mu_0 \epsilon_0 \frac{\partial \mathrm{V}}{\partial t}\Big) + \mu_0 \mathrm{J} \tag{5c}$$

In 4-vector form, the Maxwell's equations can be rewritten as :

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu, \qquad \partial_\mu G^{\mu\nu} = 0 \tag{6}$$

where, '**F**' is Field Tensor and the '**G**' is Dual Tensor of EM Wave.
And in Differential geometric form , the equations can be written in a very concise form as :

$$* \mathrm{d} * \mathrm{F} = \mathrm{J}, \qquad \mathrm{dF} = 0 \tag{7}$$

Let's dive deep into these equations and see how we derive these .

2 Maxwell's equations in Differential form

These are the Maxwell equations in Vector Algebra Notation (Differential form , to be precise).

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{8}$$

Also known as Gauss law of Electrostatics , the first equation states that divergence of Electric field ($\mathbf{E}$) at any point in space is equal to the charge density(ρ) divided by permittivity of free space (ϵ_0).

$$\nabla \cdot \mathbf{B} = 0 \tag{9}$$

Also known as Gauss law of Magnetostatics , the second equation of Maxwell states that divergence of Magnetic field ($\mathbf{B}$) at any point is zero, indicating the absence of magnetic monopoles.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10}$$

Maxwell's third equation a.k.a. Faraday's law describes how a changing magnetic field induces an electric field. The curl of the electric field ($\mathbf{E}$) is equal to the negative rate of change of the magnetic field ($\mathbf{B}$) with respect to time.

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{11}$$

This equation combines Ampère's circuital law with Maxwell's addition. It states that the curl of the magnetic field ($\mathbf{B}$) is equal to the sum of the current density ($\mathbf{J}$) and the displacement current ($\epsilon_0 \frac{\partial \mathrm{E}}{\partial t}$), where ε_0 is the permittivity of free space and μ_0 is the permeability of free space. Now we assume that there is charge-free vacuum , so equations (8) and (11) become :

$$\nabla \cdot \mathbf{E} = 0 \tag{12}$$

, and

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \tag{13}$$

Note : In all the above equations and equations that we are going to be dealing with later in this paper. The symbol ∇ , also known as *nabla* is basically : $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

3 Maxwell's equations in Potential form

Before we get into deriving the equations , we have to know that the potential form allows for the formulation of the equations in terms of scalar potential($\mathbf{V}$) and vector potential ($\mathbf{A}$), which may simplify problem-solving in certain situations.
Point to be noted that some resources use the symbol ϕ in place of $\mathbf{V}$ that we are going to be using here. So lets not get confused by this .
Thus, the electric field ($\mathbf{E}$) can be written as :

$$\mathrm{E} = -\frac{\partial \mathrm{A}}{\partial t} - \nabla \mathrm{V} \tag{14}$$

Now , using this equation in equation(8), we get ,

$$\nabla \cdot (-\frac{\partial \mathrm{A}}{\partial t} - \nabla \mathrm{V}) = \frac{\rho}{\varepsilon_0} \tag{15}$$

This gives :

$$-\nabla^2 \mathrm{V} - \nabla \cdot \frac{\partial \mathrm{A}}{\partial t} = \frac{\rho}{\varepsilon_0} \tag{16}$$

or

$$\boxed{\nabla^2 \mathrm{V} + \frac{\partial}{\partial t}(\nabla \cdot \mathrm{A}) = -\frac{\rho}{\epsilon_0}} \tag{17}$$

The magnetic field can be written only in terms of the vector potential as :

$$\mathrm{B} = \nabla \times \mathrm{A} \tag{18}$$

. Putting this in equation (9) we get ,

$$\boxed{\nabla \cdot (\nabla \times \mathrm{A}) = 0} \tag{19}$$

,which is true since divergence of a curl of a vector is zero.

Now, to derive the final equation in potential form, we will use the 4^{th} Maxwell's equation or equation(11). Using the equation(18) in equation(11) we get :

$$\nabla \times (\nabla \times \mathrm{A}) = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{20}$$

This gives :

$$\nabla(\nabla \cdot \mathrm{A}) - \nabla^2 \mathrm{A} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{21}$$

Now use equation(14) here to get :

$$\nabla(\nabla \cdot \mathrm{A}) - \nabla^2 \mathrm{A} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial \mathrm{A}}{\partial t} - \nabla \mathrm{V} \right) \right) \tag{22}$$

$$\nabla(\nabla \cdot \mathrm{A}) - \nabla^2 \mathrm{A} = \mu_0 \mathrm{J} - \mu_0 \varepsilon_0 \nabla \frac{\partial \mathrm{V}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{A}}{\partial t^2} \tag{23}$$

After rearranging , we get :

$$\boxed{-\nabla^2 \mathrm{A} + \epsilon_0 \mu_0 \frac{\partial^2 \mathrm{A}}{\partial t^2} = -\nabla \Big(\nabla \cdot \mathrm{A} + \mu_0 \epsilon_0 \frac{\partial \mathrm{V}}{\partial t} \Big) + \mu_0 \mathrm{J}} \tag{24}$$

These equations are homogenous wave equations. Applying the Lorentz gauge given as :

$$\nabla \cdot \mathrm{A} + \frac{1}{c^2} \frac{\partial \mathrm{V}}{\partial t} = 0 \tag{25}$$

in equations(17) and (24) gives us the inhomogenous wave equations given as :

$$\boxed{\frac{1}{c^2}\frac{\partial^2 \mathrm{V}}{\partial t^2} - \nabla^2 \mathrm{V} = \frac{\rho}{\varepsilon_0}} \tag{26}$$

and

$$\boxed{\frac{1}{c^2}\frac{\partial^2 \mathrm{A}}{\partial t^2} - \nabla^2 \mathrm{A} = \mu_0 \mathrm{J}} \tag{27}$$

4 Maxwell's equations in 4-vector form

4.1 Introduction to tensors

Mathematically, a tensor can be represented as a multi-dimensional array of components. Let T be a tensor with components $T^{i_1 i_2 \dots i_n}$, where $i_1, i_2, \dots, i_n$ are indices corresponding to each dimension.

$$T^{i_1 i_2 \dots i_n} \tag{28}$$

4.1.1 Covariant Tensor

A covariant tensor of rank k is denoted as $A_{i_1 i_2 \dots i_k}$. Under a coordinate transformation, the components of a covariant tensor transform according to the Jacobian matrix of the coordinate transformation.

$$A'_{i'_1 i'_2 \dots i'_k} = \frac{\partial x_{i'_1}}{\partial x_{j_1}} \frac{\partial x_{i'_2}}{\partial x_{j_2}} \cdots \frac{\partial x_{i'_k}}{\partial x_{j_k}} A_{j_1 j_2 \dots j_k} \tag{29}$$

4.1.2 Contravariant Tensor

A contravariant tensor of rank k is denoted as $B^{i_1 i_2 \dots i_k}$. Under a coordinate transformation, the components of a contravariant tensor transform with the inverse of the Jacobian matrix.

$$B'^{i'_1 i'_2 \dots i'_k} = \frac{\partial x_{j_1}}{\partial x_{i'_1}} \frac{\partial x_{j_2}}{\partial x_{i'_2}} \cdots \frac{\partial x_{j_k}}{\partial x_{i'_k}} B^{j_1 j_2 \dots j_k} \tag{30}$$

4.2 Tensors used

First of all we note that an object can be denoted by time and spatial coordinates as :

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{31}$$

Differential of this is :

$$\partial x^\mu = (\partial x^0, \partial x^1, \partial x^2, \partial x^3) = (c\partial t, \partial x, \partial y, \partial z) \tag{32}$$

For example :

$$A^\mu = (A^0, A^1, A^2, A^3) = (A_t, A_x, A_y, A_z) = (A_t, \overrightarrow{A}) \tag{33}$$

and hence the differential with respect to x^μ gives :

$$\frac{\partial A^\mu}{\partial x^\mu} = \frac{1}{c}\frac{\partial A_t}{\partial t} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{c}\frac{\partial A_t}{\partial t} + (\nabla \cdot \overrightarrow{A}) \tag{34}$$

The Electromagnetic field tensor is given as :

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{35}$$

The Dual Tensor of this is given as :

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \tag{36}$$

And the current vector in 4-form is :

$$\mathrm{J}^\mu = (c\rho, J_x, J_y, J_z) = (c\rho, \overrightarrow{\mathrm{J}}) \tag{37}$$

4.3 Derivation

To derive , first of all we have to write our Field tensor and Dual tensor in the respective forms as :

$$F^{\mu\nu} = F^{\mu\lambda} g_{\lambda\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{38}$$

and

$$G^{\mu\nu} = G^{\mu\lambda} g_{\lambda\nu} \begin{pmatrix} 0 & B_x & B_y & B_z \\ B_x & 0 & -E_z/c & E_y/c \\ B_y & E_z/c & 0 & -E_x/c \\ B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \tag{39}$$

Point to be noted is that we have multiplied the tensors with Levi-Civita symbol.

To derive the first equation we may use the first Maxwell equation (equation 8) or the last Maxwell equation (equation 11). Using equation (8) we get ,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0} \tag{40}$$

This leads to :

$$\frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} = \frac{j^0}{\varepsilon_0 c^2} = \mu_0 j^0 \tag{41}$$

or

$$\partial_\mu F^{\mu 0} = \mu_0 j^0 \tag{42}$$

Using equation (11) for x component (for showing how it works. We can use other components too) we get ,

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \frac{1}{c^2}\frac{\partial E_x}{\partial t} \tag{43}$$

rearranging this , we get :

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = \mu_0 J_x \tag{44}$$

implies ,

$$\partial_2 F^{21} + \partial_3 F^{31} + \partial_0 F^{01} = \mu_0 j^1 \tag{45}$$

Solving for other components yield similar results, and hence combining all solutions we conclude :

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu} \tag{46}$$

Now comes deriving the second equation . We will use Maxwell's second equation (equation 9) and third equation (equation 10).
So, firstly equation (9) can be written as :

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{47}$$

which can be further written in terms of the dual tensor as :

$$\frac{\partial G^{10}}{\partial x^1} + \frac{\partial G^{20}}{\partial x^2} + \frac{\partial G^{30}}{\partial x^3} = 0 \tag{48}$$

or

$$\partial_\mu G^{\mu 0} = 0 \tag{49}$$

Equation(10) can be written for x component as :

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \tag{50}$$

Multiply both sides with $\frac{1}{c}$ to get :

$$\frac{\partial}{\partial y}(\frac{E_z}{c}) + \frac{\partial}{\partial z}(-\frac{E_y}{c}) + \frac{\partial B_x}{c\partial t} = 0 \tag{51}$$

Writing this in terms of the Dual tensor we get :

$$\partial_2 G^{21} + \partial_3 G^{31} + \partial_0 G^{01} = 0 \tag{52}$$

Solving for other components yield similar results, and hence combining all solutions we conclude :

$$\boxed{\partial_\mu G^{\mu\nu} = 0} \tag{53}$$

5 Maxwell's equations in Differential Geometric form

5.1 Introduction to the Wedge Product

The wedge product of vectors result in bivector . The symbol used is $\wedge$.
For example :

$$u \wedge v = u \otimes v - v \otimes u = \begin{pmatrix} 0 & u_1v_2 - u_2v_1 & u_1v_3 - u_3v_1 \\ u_2v_1 - u_1v_2 & 0 & u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 & u_3v_2 - u_2v_3 & 0 \end{pmatrix} \tag{54}$$

where $u \otimes v$ is the outer product of u and v .

Properties of Wedge Product Operator are :

1. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
2. $(a + b) \wedge (c + d) = (a \wedge c) + (a \wedge d) + (b \wedge c) + (b \wedge d)$
3. $u \wedge v = -v \wedge u$
4. $u \wedge u = 0$
5. $*(a \wedge b) = a \times b$; $*(a \times b) = a \wedge b$.
   Here * is the Hodge Dual Operator .

5.2 Introduction the Hodge Dual Operator

The Hodge dual of a differential form ω in n dimensions is denoted by $*\omega$.
The Hodge dual operator is a mathematical tool used in differential geometry and algebraic topology. Its primary purpose is to establish a correspondence between certain types of geometric objects, such as differential forms, and their dual counterparts.

Properties of Hodge Dual Operator are :

1. $*1 = dt \wedge dx \wedge dy \wedge dz \quad ; \quad *(dt \wedge dx \wedge dy \wedge dz) = -1$
2. $*dt = dx \wedge dy \wedge dz \quad ; \quad *dx = dt \wedge dy \wedge dz$
   $*\, dy = dt \wedge dx \wedge dz \quad ; \quad *dz = dt \wedge dx \wedge dy$
3. $*(dt \wedge dx) = dz \wedge dy \quad ; \quad *(dz \wedge dy) = -dt \wedge dx$
   $*\, (dt \wedge dy) = dx \wedge dz \quad ; \quad *(dx \wedge dz) = -dt \wedge dy$
   $*\, (dt \wedge dz) = dy \wedge dx \quad ; \quad *(dy \wedge dx) = -dt \wedge dz$

5.3 Derivation

Since we know that the Electric and Magnetic fields are represented as three component form like $E =< E_1, E_2, E_3 >$ and $B =< B_1, B_2, B_3$,
We can write them in 2-form using the Wedge product as :

$$F = E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt + B_1 dy \wedge dz + B_2 dx \wedge dz + B_3 dx \wedge dy \tag{55}$$

Also, the current can be written in 1-form as :

$$\mathrm{J} = \rho dt - J_1 dx - J_2 dy - J_3 dz \tag{56}$$

Now, we take the exterior derivative of F.

$$\begin{aligned}
\mathrm{dF} = &\left(\frac{\partial E_1}{\partial y} dy \wedge dx \wedge dt + \frac{\partial E_1}{\partial z} dz \wedge dx \wedge dt\right) \\
&+ \left(\frac{\partial E_2}{\partial x} dx \wedge dy \wedge dt + \frac{\partial E_2}{\partial z} dz \wedge dy \wedge dt\right) \\
&+ \left(\frac{\partial E_3}{\partial x} dx \wedge dz \wedge dt + \frac{\partial E_3}{\partial y} dy \wedge dz \wedge dt\right) \\
&+ \left(\frac{\partial B_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial B_1}{\partial t} dt \wedge dy \wedge dz\right) \\
&+ \left(\frac{\partial B_2}{\partial y} dy \wedge dx \wedge dz + \frac{\partial B_2}{\partial t} dt \wedge dx \wedge dz\right) \\
&+ \left(\frac{\partial B_3}{\partial z} dz \wedge dx \wedge dy + \frac{\partial B_3}{\partial t} dt \wedge dx \wedge dy\right)
\end{aligned} \tag{57}$$

This gives :

$$\begin{aligned}
\mathrm{dF} = &\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t}\right) dt \wedge dx \wedge dy \\
&+ \left(\frac{\partial E_3}{\partial x} - \frac{\partial E_1}{\partial z} + \frac{\partial B_2}{\partial t}\right) dt \wedge dx \wedge dz \\
&+ \left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z} + \frac{\partial B_1}{\partial t}\right) dt \wedge dy \wedge dz \\
&+ \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right) dx \wedge dy \wedge dz
\end{aligned} \tag{58}$$

And finally,

$$\begin{aligned}
\mathrm{dF} = &\left((\nabla \times E)_3 + \frac{\partial B_3}{\partial t}\right) dt \wedge dx \wedge dy \\
&+ \left((\nabla \times E)_2 + \frac{\partial B_2}{\partial t}\right) dt \wedge dx \wedge dz \\
&+ \left((\nabla \times E)_1 + \frac{\partial B_1}{\partial t}\right) dt \wedge dy \wedge dz \\
&+ (\nabla \cdot B) dx \wedge dy \wedge dz
\end{aligned} \tag{59}$$

Using equation (9) and (10) , we get

$$\boxed{\mathrm{dF} = 0} \tag{60}$$

Now, we are to find the second equation in differential geometric form. For that , firstly we find the Hodge dual of F .

$$*\mathrm{F} = E_1 dy \wedge dz - E_2 dx \wedge dz + E_3 dx \wedge dy + B_1 dt \wedge dx - B_2 dt \wedge dy + B_3 dt \wedge dz \tag{61}$$

Now similar to what we did above , we find the exterior derivative of *F to get :

$$\begin{aligned} d * \mathrm{F} =& (\nabla \cdot \mathrm{E}) dx \wedge dy \wedge dz \\ &+ \left(\frac{\partial E_1}{\partial t} - (\nabla \times B)_1 \right) dt \wedge dy \wedge dz \\ &+ \left(\frac{\partial E_2}{\partial t} - (\nabla \times B)_2 \right) dt \wedge dx \wedge dz \\ &+ \left(\frac{\partial E_3}{\partial t} - (\nabla \times B)_3 \right) dt \wedge dx \wedge dy \end{aligned} \tag{62}$$

Using equations (8) and (11) here we get :

$$d * \mathrm{F} = \rho dx \wedge dy \wedge dz - J_1 dt \wedge dy \wedge dz - J_2 dt \wedge dx \wedge dz - J_3 dt \wedge dx \wedge dy \tag{63}$$

Again applying Hodge dual over equation above, we get,

$$*d * \mathrm{F} = \rho dt - J_1 dx - J_2 dy - J_3 dz \tag{64}$$

From equation (56), we see that RHS of equation above changes as :

$$\boxed{*\mathrm{d} * \mathrm{F} = \mathrm{J}} \tag{65}$$

6 Conclusion

In conclusion, the derived formulations of Maxwell's equations presented in this study showcase the inherent mathematical beauty and versatility of these fundamental principles in electromagnetism. The vector algebraic differential form, potential form, 4-vector form, and differential geometric form collectively illustrate the diverse mathematical expressions that encapsulate the profound nature of electromagnetic phenomena.
In essence , the derived equations not only serve as a testament to the elegance of Maxwell's theory but also highlight the interconnectedness of mathematical principles in describing the fundamental laws governing electromagnetism.